Mathematics Talent Reward Programme

Question Paper for Class XI

17^{th} January, 2016

Total Marks: 150 | Allotted Time: 10:00 a.m. to 12:30 p.m.

Multiple Choice Questions

[***You should answer these questions in the first page according to the order given in the question paper***. *Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.*]

1. Sum of roots in the range $(-\frac{\pi}{2}, \frac{\pi}{2})$ of the equation $\sin x \tan x = x^2$ is

   (A) $\frac{\pi}{2}$, **(B)** 0, **(C)** 1, **(D)** None of these.

2. Let f be a function satisfying $f(x+y+z) = f(x) + f(y) + f(z)$ for all integers x, y, z. Suppose $f(1) = 1$ and $f(2) = 2$. Then $\lim_{n\to\infty} \frac{1}{n^3} \sum_{r=1}^{n} 4r f(3r)$ equals

   (A) 4, **(B)** 6, **(C)** 12, **(D)** 24.

3. z is a complex number and $|z| = 1$ and $z^2 \neq 1$. Then $\frac{z}{1-z^2}$ lies on

   (A) a line not passing through origin, **(B)** $|z| = 2$
   (C) x-axis, **(D)** y-axis.

4. There are 168 primes below 1000. Then sum of all primes below 1000 is

   (A) 11555 , **(B)** 76127, **(C)** 57298 , **(D)** 81722.

5. ABCD is a quadrilateral on complex plane whose four vertices satisfy $z^4 + z^3 + z^2 + z + 1 = 0$. Then $ABCD$ is a

   (A) Rectangle, **(B)** Rhombus, **(C)** Isosceles Trapezium, **(D)** Square.

6. Number of solutions of the equation $3^x + 4^x = 8^x$ in reals is

   (A) 0, **(B)** 1, **(C)** 2, **(D)** ∞.

7. Let $\{x\}$ denote the fractional part of x. Then $\lim_{n\to\infty} \{(1+\sqrt{2})^{2n}\}$ equals

   (A) 0, **(B)** 0.5, **(C)** 1, **(D)** does not exist.

8. Let p be a prime such that $16p + 1$ is a perfect cube. A possible choice for p is

   (A) 283, **(B)** 307, **(C)** 593, **(D)** 691.

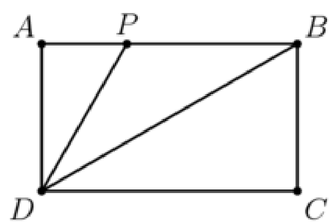
9. f be a function satisfying $2f(x) + 3f(-x) = x^2 + 5x$. Find $f(7)$.

   (A) $-\frac{105}{4}$, **(B)** $-\frac{126}{5}$, **(C)** $-\frac{120}{7}$, **(D)** $-\frac{132}{7}$.

10. Let $A = \{1, 2, \ldots, 100\}$. Let S be a subset of the power set of A such that any two elements of S has non zero intersection (Note that elements of S are actually some subsets of A). Then the maximum possible cardinality of S is

    (A) 2^{99}, **(B)** $2^{99} + 1$, **(C)** $2^{99} + 2^{98}$, **(D)** None of these.

11. In rectangle $ABCD$, $AD = 1$, P is on $\overline{AB}$, and $\overline{DB}$ and $\overline{DP}$ trisect $\angle ADC$. What is the perimeter of $\triangle BDP$?

    (A) $3 + \frac{\sqrt{3}}{3}$, **(B)** $2 + \frac{4\sqrt{3}}{3}$, **(C)** $2 + 2\sqrt{2}$, **(D)** $\frac{3+3\sqrt{5}}{2}$.

12. Let $f(x) = (x-1)(x-2)(x-3)$. Consider $g(x) = \min\{f(x), f'(x)\}$. Then the number of points of discontinuity are

    (A) 0, **(B)** 1, **(C)** 2, **(D)** more than 2.

13. Let $P(x) = x^2 + bx + c$. Suppose $P(P(1)) = P(P(-2)) = 0$ and $P(1) \neq P(-2)$. Then $P(0) =$

    (A) $-\frac{5}{2}$, **(B)** $-\frac{3}{2}$, **(C)** $-\frac{7}{4}$, **(D)** $\frac{6}{7}$.

14. Let $[x]$ denotes the greatest integer less than or equal to x. Find x such that $x[x[x[x]]] = 88$

(A) π, **(B)** 3.14, **(C)** $\frac{22}{7}$, **(D)** All of these.

15. Suppose $50x$ is divisible by 100 and kx is not divisible by 100 for all $k = 1, 2, \ldots, 49$. Find number of solutions for x when x takes values $1, 2, \ldots, 100$.

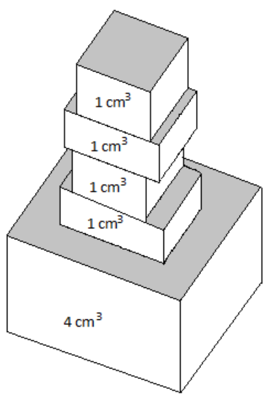
(A) 20, **(B)** 25, **(C)** 15, **(D)** 50.

Short Answer Type Questions

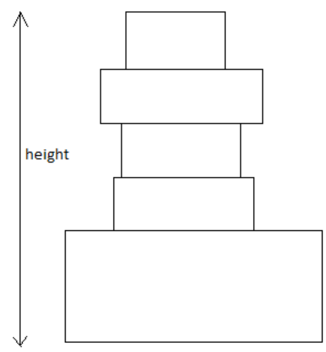
[*Each question carries a total of 15 marks. Credits will be given to partially correct answers*]

1. Show that there exist a polynomial $P(x)$ whose one coefficient is $\frac{1}{2016}$ and remaining coefficients are rational numbers, such that $P(x)$ is an integer for any integer x.

2. 5 blocks of volume 1 cm^3,1 cm^3,1 cm^3,1 cm^3, and 4 cm^3 are placed one above another to form the structure as shown in the figure. Suppose the sum of surface areas of **upper face** of each block is 48 cm^2. Determine the minimum possible height of the whole structure.



Structure



Front view of the structure

3. Prove that for any positive integer n there are n consecutive composite numbers all less than 4^{n+2}. [You may use the fact that product of all primes, which are less than k, is less than 4^k and this holds for all positive integers k.]

4. For any given k points in a plane, we define the diameter of the points as the maximum distance between any two points among the given points. Suppose n point are there in a plane with diameter d. Show that we can always find a circle with radius $\frac{\sqrt{3}}{2}d$ such that all the points lie inside the circle.

5. Let $\mathbb{N}$ be the set of all positive integers. $f, g : \mathbb{N} \to \mathbb{N}$ be functions such that f is onto, g is one-one and $f(n) \geq g(n)$ for all positive integers n. Prove that $f = g$.

6. Consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. A partition Π of A is a collection of disjoint sets whose union is A. For example, $\Pi_1 = \{\{1, 2\}, \{3, 4, 5\}, \{6, 7, 8, 9\}\}$ and $\Pi_2 = \{\{1\}, \{2, 5\}, \{3, 7\}, \{4, 6, 8, 9\}\}$ can be considered as partitions of A. For each Π partition, we consider the function π defined on the elements of A. $\pi(x)$ denotes the cardinality of the subset in Π which contains x. For example, in case Π_1, $\pi_1(1) = \pi_1(2) = 2, \pi_1(3) = \pi_1(4) = \pi_1(5) = 3$, and $\pi_1(6) = \pi_1(7) = \pi_1(8) = \pi_1(9) = 4$. For Π_2 we have, $\pi_2(1) = 1$, $\pi_2(2) = \pi_2(5) = 2, \pi_2(3) = \pi_2(7) = 2$, and $\pi_2(4) = \pi_2(6) = \pi_2(8) = \pi_2(9) = 4$. Given any two partitions Π and Π', show that there are two numbers x and y in A, such that $\pi(x) = \pi'(x)$ and $\pi(y) = \pi'(y)$. [Hint: Consider the case where there is a block of size greater than or equal to 4 in a partition and the alternative case.]

Use of calculators is not allowed. You may use a ruler and a compass for construction.

~ Best of Luck ~

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