Mathematics Talent Reward Programme

Model Solutions for Class XI

Multiple Choice Questions

[Each question has only one correct option. You will be awarded 4 marks for the correct answer, 1 mark if the question is not attempted and 0 marks for wrong answer.]

1. **(B)**	2. **(A)**	3. **(D)**	4. **(B)**	5. **(C)**
6. **(B)**	7. **(C)**	8. **(B)**	9. **(B)**	10. **(A)**
11. **(B)**	12. **(A)**	13. **(A)**	14. **(C)**	15. **(A)**

Short Answer Type Questions

[Each question carries a total of 15 marks. Credit will be given to partially correct answers]

1. Consider $P(x) = \frac{1}{2016}(x-1)(x-2)\cdots(x-2016)$. Clearly all coefficients of $P(x)$ are rationals. Observe that the leading coefficient of $P(x)$ is $\frac{1}{2016}$. Note that product of 2016 consecutive integers is always divisible by 2016. Hence this $P(x)$ is our required polynomial.

2. First Solution: Let $a_1, a_2, \ldots, a_5$ be the surface areas of upper face of the blocks. We have $a_1 + a_2 + \cdots + a_5 = 48$. Note that the heights of the blocks are given by

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}, \quad \text{and} \quad \frac{4}{a_5}$$

Note that by Cauchy-Schwarz inequality

$$\begin{aligned} &48\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{4}{a_5}\right)\\ &=\left(\frac{1}{(\sqrt{a_1})^2}+\frac{1}{(\sqrt{a_2})^2}+\frac{1}{(\sqrt{a_3})^2}+\frac{1}{(\sqrt{a_4})^2}+\left(\frac{2}{\sqrt{a_5}}\right)^2\right)\left((\sqrt{a_1})^2+\cdots+(\sqrt{a_5})^2\right)\\ &\geq (1+1+1+1+2)^2 = 36 \end{aligned}$$

Hence

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{4}{a_5} \geq \frac{36}{48} = \frac{3}{4}$$

Equality holds when $a_5 = 2a_1 = 2a_2 = 2a_3 = 2a_4 = 16$. Thus minimum possible height is certainly $\frac{3}{4}$.

Second solution: One can arrive at the same conclusion using AM-HM inequality. Observe that

$$\begin{aligned} &\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{4}{a_5}\\ &=\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{1}{a_5/2}+\frac{1}{a_5/2}\\ &\geq \frac{36}{a_1+a_2+a_3+a_4+a_5/2+a_5/2} = \frac{36}{48} = \frac{3}{4} \end{aligned}$$

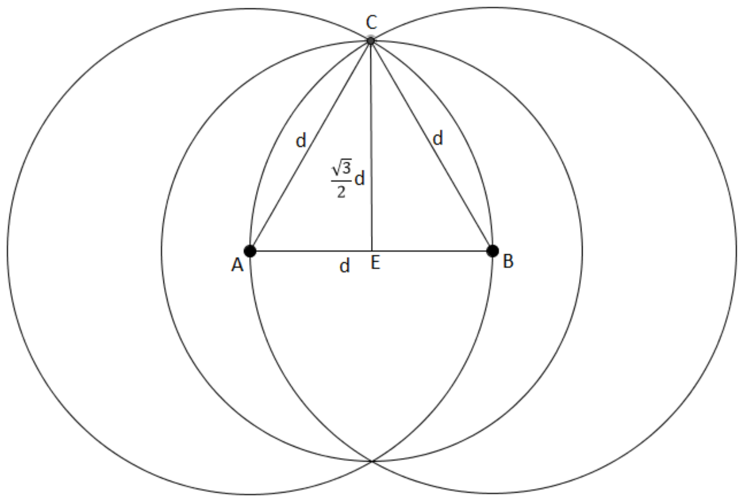
□

3. Let P be the product of all primes which are less than or equal to $n+1$. Consider the following n consecutive numbers

$$P+2, P+3, \ldots, P+(n+1)$$

We will now show that $P+k$ is composite for any $k \in \{2, 3, \ldots n+1\}$. Consider any prime divisor of k say u. Clearly u divides P. Thus u divides $P+k$ implying $P+k$ is composite. This completes the proof. □

4. Consider the two points say A and B whose distance is d. Now all points must lie within the circle centre at A as well as the circle centre at B. So they must lie within the intersection of the two circles. Consider the circle drawn with centre as the mid point of A and B and with radius the altitude of the shown equilateral triangle ABC. Note that the radius equals $\frac{\sqrt{3}}{2}d$. This new circle covers the intersection region of the two circles and hence all the points within it.



5. We shall prove the statement $P(k)$: For every $k \in \mathbb{N}$ there exists a unique $x_k \in \mathbb{N}$ such that $f(x_k) = g(x_k) = k$ by induction on k. Since f is onto, there exists $x_1 \in \mathbb{N}$ such that $f(x_1) = 1$. But $g \leq f$ so $g(x_1) = 1$. Since g is one-one this x_1 is unique. Thus we have proved $P(1)$. Now let $P(k)$ be true. We shall prove that $P(k+1)$ is true. As f is onto, there exists $x_{k+1} \in \mathbb{N}$ such that $f(x_{k+1}) = k + 1$. But $g \leq f$ and by induction hypothesis g already takes all values less that $k + 1$. So $g(x_{k+1}) = k + 1$. Since g is one-one this x_{k+1} is unique. Thus by the principle of mathematical induction, the statement $P(k)$ holds for all natural numbers k. Observe that $P(k)$ implies $f = g$. This completes the proof. □

6. Part 1: Let us assume for the sake of contradiction there do not exist two distinct elements satisfying the property.

   Part 2: If there were 4 distinct values of π or π' in decreasing, the first value is at least 4, the second value is at least 3, the third value is at least 2, and the fourth value is at least 1, so we need at least 4 elements in the first partition, 3 in the second partition, 2 in the third partition, and 1 in the fourth partition, which implies we need at least 10 elements. But $4+3+2+1 > 9$. Hence there can only be at most 3 distinct values for $\pi(x)$ and $\pi'(x)$.

   Part 3: In addition, only at most three elements can share a value for $\pi(x)$ or $\pi'(x)$ If there were 4 elements which had the same value for $\pi(x)$, WLOG $\pi(1) = \pi(2) = \pi(3) = \pi(4)$, then by Part 1 we know $\pi'(1)$, $\pi'(2)$, $\pi'(3)$, $\pi'(4)$ must all be different, but this contradicts Part 2. So for each value of $\pi(x)$, only at most three elements can share that value.

   So we know from Part 3 that $\pi(x)$ cannot equal 4. It follows that the only possible way is for 3 elements to satisfy $\pi(x) = 3$, 3 other elements to satisfy $\pi(x) = 2$, and 3 other elements to satisfy $\pi(x) = 1$. But it's impossible for exactly three elements to satisfy $\pi(x) = 2$, because the number of elements satisfying $\pi(x) = 2$ must be even. So we have a contradiction. □