Mathematics Talent Reward Programme

Question Paper for Class XI

15^{th} January, 2017

Total Marks: 102 Allotted Time: 10:00 a.m. to 12:30 p.m.

Multiple Choice Questions

[***You should answer these questions in the first page according to the order given in the question paper***. *Each question has only one correct option. You will be awarded 3 marks for the correct answer, 0 marks if the question is not attempted and -1 mark for wrong answer.*]

1. The number of real solutions of the equation $(9/10)^x = -3 + x - x^2$ is

   (A) 2, **(B)** 0, **(C)** 1, **(D)** None of these.

2. $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} =$

   (A) $\sqrt{e}$, **(B)** ∞, **(C)** Does not exist, **(D)** None of these.

3. Let $p(x) = x^4 - 4x^3 + 2x^2 + ax + b$. Suppose that for every root λ of p, $1/\lambda$ is also a root of p. Then $a + b =$

   (A) -3, **(B)** -6, **(C)** -4, **(D)** -8.

4. Let $F_1 = F_2 = 1$. We define inductively $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 2$. Then the sum

   $$F_1 + F_2 + F_3 + \cdots + F_{2017}$$

   is

   (A) even but not divisible by 3, **(B)** odd but divisible by 3
   (C) odd and leaves remainder 1 when divisible by 3, **(D)** None of these.

5. Compute the number of ordered quadruples of positive integers (a, b, c, d) such that

   $$a! \cdot b! \cdot c! \cdot d! = 24!$$

   (A) 4, **(B)** 4!, **(C)** 4^4, **(D)** None of these .

6. Let $p(x)$ is a polynomial of degree 4 with leading coefficients 1. Suppose $p(1) = 1, p(2) = 2, p(3) = 3$ and $p(4) = 4$. Then $p(5) =$

   (A) 5, **(B)** $\frac{25}{6}$, **(C)** 29, **(D)** 35.

7. Let $ABCD$ be a quadrilateral with sides $AB = 2, BC = CD = 4$ and $DA = 5$. The opposite angles A and C are equal. The length of diagonal BD equals

   (A) $2\sqrt{6}$, **(B)** $3\sqrt{3}$, **(C)** $3\sqrt{6}$, **(D)** $2\sqrt{3}$.

8. How many finite sequences $x_1, x_2, \ldots, x_m$ are there such that $x_i = 1$ or 2 and $\sum_{i=1}^{m} x_i = 10$?

   (A) 89, **(B)** 73, **(C)** 107, **(D)** 119.

9. From a point P outside of a circle with centre O, tangent segments PA and PB are drawn. If $\frac{1}{OA^2} + \frac{1}{PA^2} = \frac{1}{16}$. Then $AB =$

   (A) 4, **(B)** 6, **(C)** 8, **(D)** 10.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $\lim_{x\to\infty} f'(x) = 1$, then

    (A) f is increasing, **(B)** f is unbounded, **(C)** f' is bounded, **(D)** All of these.

Short Answer Type Questions

[*Each question carries a total of 12 marks. Credits will be given to partially correct answers*]

1. A monic polynomial is a polynomial whose highest degree coefficient is 1. Let $P(x)$ and $Q(x)$ be monic polynomials with real coefficients, and $\deg P(x) = \deg Q(x) = 10$. Prove that if the equation $P(x) = Q(x)$ has no real solutions, then $P(x + 1) = Q(x - 1)$ has a real solution.

2. Let a, b, c be positive reals such that $a + b + c = 3$. Show that

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} \leq \frac{6}{\sqrt{(a+b)(b+c)(c+a)}}$$

3. Let $f : [0, 1] \to [0, 1]$ be a continuous function. We say $f \equiv 0$ if $f(x) = 0$ for all $x \in [0, 1]$ and similarly $f \not\equiv 0$ if there exist at least one $x \in [0, 1]$ such that $f(x) \neq 0$. Suppose $f \not\equiv 0$, $f \circ f \not\equiv 0$, but $f \circ f \circ f \equiv 0$. Do there exist such an f? If yes construct such an function, if no prove it. [Note that $f \circ f(x) = f(f(x))$ and $f \circ f \circ f(x) = f(f(f(x)))$.]

4. An irreducible polynomial is a non-constant polynomial that cannot be factored into the product of two non-constant polynomials. Consider the following statements:

   Statement 1: $p(x)$ be any monic irreducible polynomial with integer coefficients and degree ≥ 4. Then $p(n)$ is prime for at least one natural number n.

   Statement 2: $n^2 + 1$ is prime for infinitely many values of natural number n.

   Show that if Statement 1 is true then Statement 2 is also true.

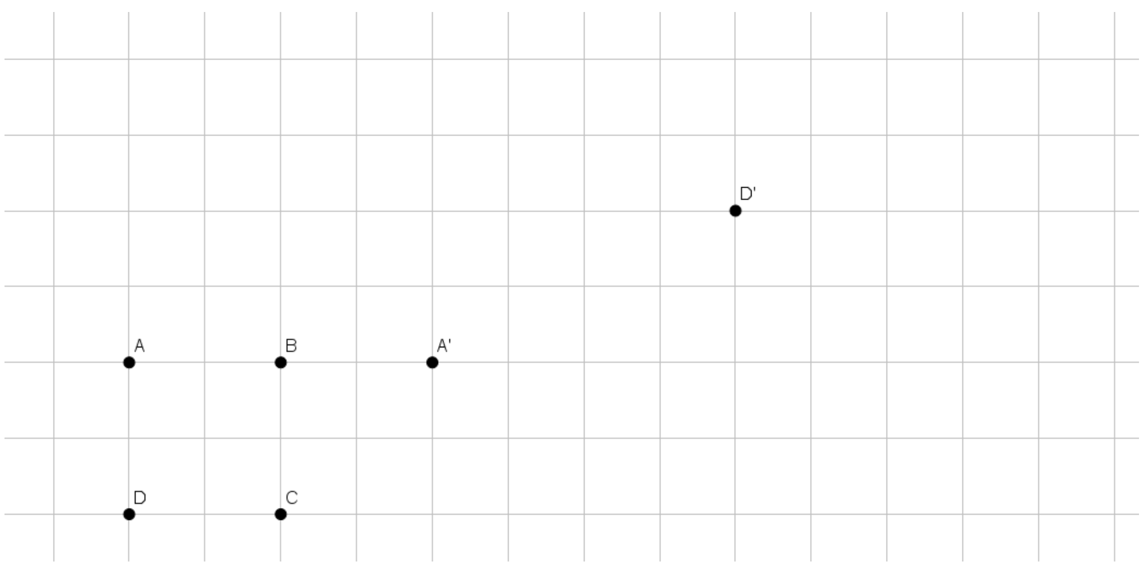
5. Let $\mathbb{N}$ be the set of all natural numbers. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijective function. Show that there exist three numbers a, b, c in arithmetic progression such that $f(a) < f(b) < f(c)$.

6. Let us consider an infinite grid plane as shown below. We start with 4 points A, B, C, D, that form a square, as shown below.

   We perform the following operation: We pick two points say X and Y from the current points. X is reflected about Y to get X'. We remove X and add X' to get a new set of 4 points and treat it as our current points.

   For example in the figure suppose we choose A and B (we can choose any other pair too). Then reflect A about B to get A'. We remove A and add A' Thus A', B, C, D is our new 4 points. We may again choose D and A' from the current points. Reflect D about A to obtain D' and hence A', B, C, D' are now new set of points. Then similar operation is performed on this new 4 points and so on.

   Starting with A, B, C, D, can you get a bigger square by some sequence of such operations?



Use of calculators is not allowed. You may use a ruler and a compass for construction.

~ Best of Luck ~