Mathematics Talent Reward Programme

Model Solutions for Class XI

Multiple Choice Questions

[*Each question has only one correct option. You will be awarded 3 marks for the correct answer, 0 marks if the question is not attempted and -1 mark for wrong answer.*]

1. **(B)**	2. **(D)**	3. **(A)**	4. **(C)**	5. **(D)**
6. **(C)**	7. **(A)**	8. **(A)**	9. **(C)**	10. **(B)**

Short Answer Type Questions

[*Each question carries a total of 12 marks. Credit will be given to partially correct answers*]

1. Let $P(x) = x^{10} + a_9x^9 + a_8x^8 + \cdots + a_0$ and $Q(x) = x^{10} + b_9x^9 + b_8x^8 + \cdots + b_0$. Let $R(x) = P(x) - Q(x)$ Note that the equation $R(x) = (a_9 - b_9)x^9 + (a_8 - b_8)x^8 + \cdots + (a_0 - b_0)$

   If $a_9 \neq b_9$, then $R(x)$ is of degree 9, then the polynomial $R(x)$ must have a real root which contradicts the assumption that $R(x) = P(x) - Q(x) = 0$ has no real solutions. Thus $a_9 = b_9$.

   Let $S(x) = P(x + 1) - Q(x - 1)$. Then

   $$S(x) = (x + 1)^{10} - (x - 1)^{10} + a_9(x + 1)^9 - a_9(x - 1)^9 + T(x)$$

   where $T(x)$ is polynomial of degree at most 8. Clearly $a_9[(x+1)^9 - (x-1)^9]$ is of degree atmost 8 since on expansion x^9 coefficient cancels out, whereas

   $$\begin{aligned}(x + 1)^{10} - (x - 1)^{10} &= [x^{10} + 10x^9 + A(x)] - [x^{10} - 10x^9 + B(x)] \\ &= 20x^9 + A(x) - B(x)\end{aligned}$$

   where $A(x)$ and $B(x)$ are polynomials of degree at most 8. Hence $S(x)$ is of degree exactly equal to 9 hence it must have a real root. Thus $P(x + 1) - Q(x - 1)$ has real solution. □

2. Note that the given inequality can be written as

   $$\sqrt{a(a + b)(a + c)} + \sqrt{b(b + c)(b + a)} + \sqrt{c(c + a)(c + b)} \leq 6$$

   Note that

   $$\begin{aligned}(a + b + c)^2 - 3(ab + bc + ca) &= a^2 + b^2 + c^2 - ab - bc - ca \\ &= \frac{1}{2}(a - b)^2 + \frac{1}{2}(b - c)^2 + \frac{1}{2}(c - a)^2 \geq 0\end{aligned}$$

   Hence $(a + b + c)^2 \geq 3(ab + bc + ca)$. Since $a + b + c = 3$, we have that $ab + bc + ca \leq 3$.

   Solution 1:

   By Cauchy Schwarz inequality we have that

   $$(a + b + c)((3a + bc) + (3b + ca) + (3c + ab)) \geq (\sqrt{a(3a + bc)} + \sqrt{b(3b + ca)} + \sqrt{c(3c + ab)})^2$$

   Note that $a+b+c = 3$ and hence $3a+bc = (a+b+c)a+bc = (a+b)(a+c)$, $3b+ca = (b+c)(b+a)$, and $3c + ab = (c + a)(c + b)$. Thus taking square roots in the above inequality we have

   $$\sqrt{3(3a + 3b + 3c + ab + bc + ca)} \geq \sqrt{a(a + b)(a + c)} + \sqrt{b(b + c)(b + a)} + \sqrt{c(c + a)(c + b)}$$

   Note that $ab + bc + ca \leq 3$, hence

   $$\sqrt{a(a + b)(a + c)} + \sqrt{b(b + c)(b + a)} + \sqrt{c(c + a)(c + b)} \leq \sqrt{3(3 \times 3 + 3)} = \sqrt{36} = 6$$

Solution 2: Observe that by AM-GM inequality we have

$$\frac{7a+bc}{2} = \frac{4a+(3a+bc)}{2} \geq \sqrt{4a(3a+bc)}$$

Since $a+b+c=3$, $3a+bc=(a+b+c)a+bc=(a+b)(a+c)$. Hence $\sqrt{4a(a+b)(a+c)} \leq \frac{1}{2}(7a+bc)$. We divide both sides by 2 to obtain

$$\sqrt{a(a+b)(a+c)} \leq \frac{1}{4}(7a+bc)$$

Analogously we obtain

$$\sqrt{b(b+c)(b+a)} \leq \frac{1}{4}(7b+ca)$$

$$\sqrt{c(c+a)(c+b)} \leq \frac{1}{4}(7c+ab)$$

Adding all three inequalities we have

$$\begin{aligned}
&\sqrt{a(a+b)(a+c)} + \sqrt{b(b+c)(b+a)} + \sqrt{c(c+a)(c+b)} \\
&\leq \frac{1}{4}(7a+bc) + \frac{1}{4}(7b+ca) + \frac{1}{4}(7c+ab) \\
&= \frac{1}{4}(7(a+b+c)+(ab+bc+ca)) \\
&\leq \frac{1}{4}(7 \times 3 + 3) = 6
\end{aligned}$$

The last inequality is due to the fact that $a+b+c=3$ and $ab+bc+ca \leq 3$. □

3. There exists such a function satisfying all conditions. We construct one such function.

   We first show how to get an f such that $f \not\equiv 0$ but $f \circ f \equiv 0$ Let $A = \{x \in [0,1] \mid f(x) = 0\}$ and $B = \{x \in [0,1] \mid f(x) \neq 0\}$ be the set where f takes value non zero. Since $f(f(x))$ is zero for all x, $f(x)$ must take values in A. If we take $A = [0, 1/2]$, we have to ensure that f is continuous, $f(x) > 0$ for all $x > 1/2$ and $f(x) \leq \frac{1}{2}$ for all x. To do this we may take f as a part of a line whose slope is sufficiently small so that $f(x) \leq \frac{1}{2}$ for all x. For example we may take $f(x) = (x - \frac{1}{2})$ for $x \geq \frac{1}{2}$. Note that continuity is maintained and $f \not\equiv 0$ and $f(x) \leq \frac{1}{2}$ for all x. Hence $f \circ f \equiv 0$.

   To do the part where $f \not\equiv 0, f \circ f \not\equiv 0$ but $f \circ f \circ f \equiv 0$, we apply the same idea, however the time we increase the slope so that $f \circ f \not\equiv 0$. Suppose $f(x) = 0$ for $x \leq 1/2$ and $f(x) = k(x - \frac{1}{2})$ for $x \geq 1/2$ where $1 < k < 2$. Then $f(f(1)) = f(k/2) \neq 0$ as $\frac{k}{2} \geq \frac{1}{2}$. So $f \circ f \not\equiv 0$. To ensure $f \circ f \circ f \equiv 0$, we note that f is a non decreasing function hence it attains maximum at $x = 1$. But $f(f(1)) = f(k/2) = \frac{1}{2}k(k-1)$ which is less than $\frac{1}{2}$ as long as we choose sufficiently close to 1. We may choose $k = \frac{3}{2}$ for example for this purpose. Then $f(f(1)) = \frac{3}{8} < \frac{1}{2}$. Then $f \circ f$ is always less than $\frac{1}{2}$ which forces $f \circ f \circ f \equiv 0$. So the following functions works:

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2] \\ \frac{3}{2}(x - \frac{1}{2}) & \text{if } x \in (1/2, 1] \end{cases}$$

□

4. Suppose statement 1 is true but statement 2 is false. Hence n^2+1 is prime for finitely many values of n. Hence there exist $N > 0$ such that n^2+1 is composite for all $n > N$. Let $p(x) = x^4 + 1$. It is easy to check that $p(x)$ is irreducible. Note that this implies $p(x+k)$ is irreducible for any k. Finally we consider $p(x+N) = (x+N)^4 + 1$. Note that statement 2 implies $p(n+N)$ is prime at least for one natural number n. Hence suppose $p(n_0+N)$ is prime for some natural number n_0, But $p(n_0+N) = ((n_0+N)^2)^2 + 1$ and $(n_0+N)^2 > N^2 \geq N$ which is a contradiction to the fact that n^2+1 is always composite after $n > N$. □

5. Since f is bijection let us choose a such that $f(a) = 1$. If $f(a) < f(a+1) < f(a+2)$ we are done. If not the only other possibility is $f(a) < f(a+2) < f(a+1)$. This implies $f(a+2)$ lies between $f(a+1)$ and $f(a)$. We then consider $a, a+2, a+4$. If $f(a) < f(a+2) < f(a+4)$ we are done. If not the only other possibility is $f(a) < f(a+4) < f(a+2)$. Since $f(a+2) < f(a+1)$, $f(a+4)$ lies between $f(a+1)$ and $f(a)$. We then consider $a, a+4, a+8$. If $f(a) < f(a+4) < f(a+8)$ does not hold, we can again conclude that $f(a+8)$ lies between $f(a+1)$ and $f(a)$ and so on.

   Note that there are finitely many natural numbers between $f(a+1)$ and 1. Since f is a bijection, only for finitely many values of n, $f(n)$ lies between $f(a+1)$ and 1. So if we continue in the above fashion we must get a n_0 such that $f(a) < f(a+2^{n_0}) < f(a+2^{n_0+1})$. □

6. Let $ABCD$ be the initial square. Suppose it is possible to reach a bigger square say $EFGH$. Note that it is not necessary that sides of $EFGH$ is parallel to the grid lines.

   We claim that the operations are reversible i.e., if starting from P, Q, R, S you reach P', Q', R', S' then you can come back to P, Q, R, S by some sequence of operations. To prove this, consider two points X and Y. Suppose we reflect X about Y to get X'. Then according to the rule we remove X and add X'. We can now reflect X' about Y to get back X. We may remove X' and add X to get back the two points. Thus such an operation is reversible. Clearly for a sequence of operations, we may apply the reversibility of each operations to get the reversed sequence of operations. This proves our claim.

   So there is a sequence of operations by which starting from $EFGH$ one can reach smaller square $ABCD$. Now without loss of generality assume $E = (0,0), F = (0,1), G = (1,1)$ and $H = (1,0)$.

   We now claim that every point that can arise by this operation has integer co-ordinates. This is true because if $X = (x_1, x_2)$ and $Y = (y_1, y_2)$, then $X' = (2y_1 - x_1, 2y_2 - x_2)$. So inductively every such point must have integer co-ordinates.

   $ABCD$ is smaller square then $EFGH$. According to the co-ordinate system that we impose, $EFGH$ has side length 1. So, $ABCD$ has side length less than 1. But A, B, C, D must have all integer co-ordinates. Since any two distinct points having integer co-ordinates is at least 1 distance apart, this gives us a contradiction. □