Mathematics Talent Reward Programme

Question Paper for Class IX

15^{th} January, 2017

Total Marks: 102 Allotted Time: 2:00 p.m. to 4:30 p.m.

Multiple Choice Questions

[***You should answer these questions in the first page according to the order given in the question****. Each question has only one correct option. You will be awarded 3 marks for the correct answer, 0 marks if the question is not attempted and -1 mark for wrong answer.*]

1. The number of ordered pairs (a, b) of natural numbers such that $a^b + b^a = 100$ is

   (A) 1 , **(B)** 2 , **(C)** 3, **(D)** 4.

2. $ABCD$ be a rectangle. E and F are the midpoints of BC and CD respectively. The area of $\triangle AEF$ is 3 sq units. The area of rectangle $ABCD$ is

   (A) 4 , **(B)** 6, **(C)** 8, **(D)** 16.

3. Suppose a, b, c are three distinct integers from 2 to 10 (both inclusive). Exactly one of ab, bc and ca is odd and abc is a multiple of 4. The arithmetic mean of a and b is an integer and so is the arithmetic mean of a, b and c. How many such (unordered) triplets are possible?

   (A) 4, **(B)** 5, **(C)** 6, **(D)** 7.

4. $PQRS$ is a rectangle in which $PQ = 2016PS$. T an U are the midpoints of PS and PQ respectively. QT an US intersect at V. Suppose

$$R = \frac{\text{Area of triangle PQT}}{\text{Area of quadrilateral QRSV}}$$

   $R =$

   (A) $\frac{5}{12}$, **(B)** $\frac{2016}{2017}$, **(C)** $\frac{2}{7}$, **(D)** $\frac{3}{8}$.

5. For any three real numbers a, b, and c, with $b \neq c$, the operation $\otimes$ is defined by:

$$\otimes(a, b, c) = \frac{a}{b - c}$$

   What is $\otimes(\otimes(1, 2, 3), \otimes(2, 3, 1), \otimes(3, 1, 2))$?

   (A) $-\frac{1}{2}$, **(B)** $-\frac{1}{4}$, **(C)** $\frac{1}{2}$, **(D)** $\frac{1}{4}$.

6. A company sells peanut butter in cylindrical jars. Marketing research suggests that using wider jars will increase sales. If the diameter of the jars is increased by 25% without altering the volume, by what percent must the height be decreased?

   (A) 10%, **(B)** 25%, **(C)** 36%, **(D)** 64%.

7. Let

$$V_1 = \frac{7^2 + 8^2 + 15^2 + 23^2}{4} - \left(\frac{7 + 8 + 15 + 23}{4}\right)^2$$

$$V_2 = \frac{6^2 + 8^2 + 15^2 + 24^2}{4} - \left(\frac{6 + 8 + 15 + 24}{4}\right)^2$$

$$V_3 = \frac{5^2 + 8^2 + 15^2 + 25^2}{4} - \left(\frac{5 + 8 + 15 + 25}{4}\right)^2$$

   Then

   (A) $V_3 < V_2 < V_1$, **(B)** $V_3 < V_1 < V_2$, **(C)** $V_1 < V_2 < V_3$, **(D)** $V_2 < V_3 < V_1$.

8. How many natural numbers, less than 2017, are divisible by 3 but not by 5?

   (A) 548, **(B)** 538, **(C)** 528, **(D)** None of these.

9. Consider 3 numbers, $4, 6$ and 10. In 1st step we choose any a, b from the 3 numbers and replace them with $\frac{3a-4b}{5}$ and $\frac{4a+3b}{5}$ to get a new triplet of numbers and again perform the operation on new triplet and so on. How many distinct ways are there to obtain $4, 6$ and 12 as a triplet for the first time?

   (A) 3, **(B)** 5, **(C)** 7, **(D)** None of these.

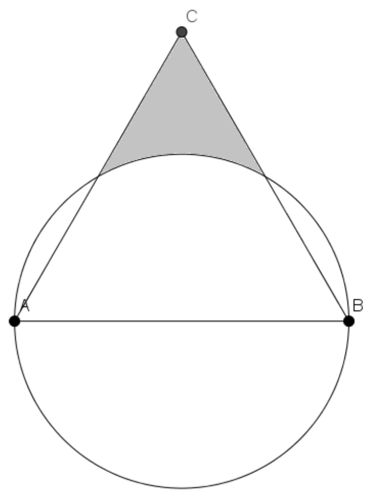
10. Let a and b be relatively prime integers with $a > b > 0$ and $\dfrac{a^3 - b^3}{(a-b)^3} = \dfrac{73}{3}$. What is $a - b$?

(A) 1, **(B)** 3, **(C)** 9, **(D)** 27.

Short Answer Type Questions

[*Each question carries a total of 12 marks. Credit will be given to partially correct answers*]

1. Let ABC be an equilateral triangle constructed on the diameter AB of circle of radius 1 as a side. Find the area of the shaded portion with justification.

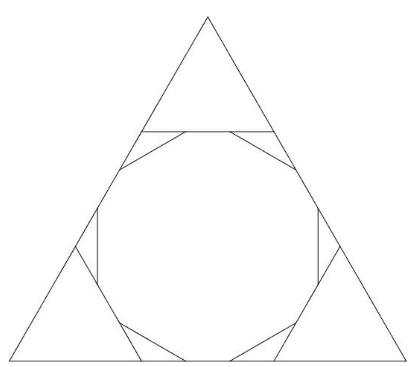


2. There are 30 balls in a box. You have to write one number in each ball. However the only numbers you are allowed to write are 0, 1 or 4. Let X be the number obtained by adding all the numbers on the balls. Find all possible values of X with justification.

3. Find all primes p and q such that $p + q = (p - q)^3$. Justify your answer.

4. The natural number y is obtained from the number x by rearranging its digits. Suppose $x + y = 10^{200}$. Prove that x is divisible by 10.

5. Consider an equilateral triangle of area 1. We call the triangle P_0. We find the trisecting points of each side of P_0 and cutoff the corners to form a new polygon (in fact a hexagon) say P_1 as shown in figure. We again trisect each side of the hexagon and cutoff the corners to form polygon P_2, with 12 sides, as shown in the figure. Find the area of P_2.



6. The numbers $1, 3, 5, 7, 2, 4, 6, 8$ are written in a row on a blackboard (in the given order). Two players A and B play the following game by making moves. In each move, a player picks two **consecutive** numbers written in the board, say a and b, and replace it by $a + b$ or $a - b$ or $a \times b$. Note that after each move there is one less number on the blackboard. Suppose player A makes the first move. The first player wins if the final result after 7 moves is odd, and loses otherwise. Show that no matter what player 1 does, player 2 can always win i.e., player 2 has a winning strategy.

Use of calculators is not allowed. You may use a ruler and a compass for construction.

$\sim$ *Best of Luck* $\sim$