# Mathematics Talent Reward Programme

Model Solutions for Class IX

### Multiple Choice Questions

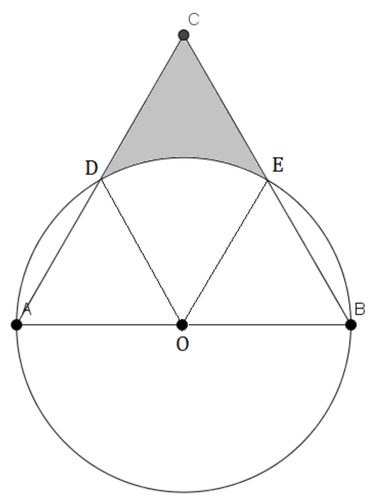
[*Each question has only one correct option. You will be awarded 3 marks for the correct answer, 0 marks if the question is not attempted and -1 mark for wrong answer.*]

| | | | | |
|---|---|---|---|---|
| 1. **(D)** | 2. **(C)** | 3. **(A)** | 4. **(D)** | 5. **(B)** |
| 6. **(C)** | 7. **(C)** | 8. **(B)** | 9. **(D)** | 10. **(B)** |

### Short Answer Type Questions

[*Each question carries a total of 15 marks. Credit will be given to partially correct answers*]

1. let us draw the center $O$ and join $OD$ and $OE$ as shown in figure.




Area $ABC = \sqrt{3}$ Note that $OD = OA$ as they are radii of the same cirle and $\angle DAO = 60°$ Hence $OAD$ is equilateral triangle with side length 1. Hence Area $DAO = \frac{\sqrt{3}}{4}$. Similarly Area $EBO = \frac{\sqrt{3}}{4}$. Note $\angle DOE = 60°$. Hence the area of sector $DOE$ is $\frac{\pi}{6}$. So the area of the shaded region is $\sqrt{3} - \frac{\sqrt{3}}{2} - \frac{\pi}{6} = \boxed{\frac{\sqrt{3}}{2} - \frac{\pi}{6}}$. □

□

2. We denote the triplet $(x, y, z)$ to denote that there are $x$ number of four-balls $y$ number of one-balls and $z$ number of zero-balls. We first show that $X$ can take any integer value from 0 to 111. Take any number from 0 to 111 say $t$. We write it as $t = 4q + r$ where $r = 0, 1, 2, 3$. Note that if $(q, r, 30 - q - r)$ is observed, $X$ can indeed achieve $t$. We just have to ensure $q, r, 30 - q - r$ are all non-negative numbers. Note that $r \leq 3$ and since $t \leq 111$, it implies $q \leq 27$. Thus $30 - q - r \geq 0$. Hence $X$ can take any integer value from 0 to 111. Note that any multiple of 4 say $4q$, less than or equal to 120, is possible if $(q, 0, 30 - q)$ observed. We note that $X = 113$ corresponds to $(28, 1, 1)$, $X = 114$ to $(28, 2, 0)$ and $X = 117$ to $(29, 1, 0)$.

   We will now show that $X$ cannot take values $115, 118, 119$. If $X = 115$ corresponds to $(x, y, z)$, then $x+y+z = 30$ and $115 = 4x+y \leq 4x+y+z = 3x+30$ which implies $3x \geq 85 \implies x \geq 29$ as $x$ is an integer, but then $4x+y \geq 4x \geq 116$, so $X = 115$ is not possible. Similarly if $X \geq 118$,

$$118 \leq X = 4x + y \leq 3x + 30 \implies 3x \geq 88 \implies x \geq 30$$

   Hence it forces $X = 120$. Hence $X$ can take any integer values between 0 to 120 except $115, 118$ and 119.

3. Suppose $p, q$ leaves same remainder when divided by 3. Then $p-q$ is divisible by 3. But $p+q$ is not divisible by 3 unless both $p, q$ are divisible by 3 which forces $p=q=3$ (as they are primes). This clearly does not gives us a solution. Thus $p, q$ leaves different remainders when divisible by 3. If both are not 3, then one of them leaves remainder 1 and the other leaves 2 when divided by 3. Then $p+q$ is divisible by 3 but $p-q$ does not. Hence no solution is possible. This forces that one of them must be 3. Clearly $p=3$ implies $q<3$ as $(3-q)^3 = 3+q > 0$. But this forces $q=2$ which does not satisfies the equation. Hence $q=3$. Thus the equation becomes

$$p+3 = (p-3)^3 = p^3 - 9p^2 + 27p - 27 \implies p(p^2 - 9p + 26) = 30$$

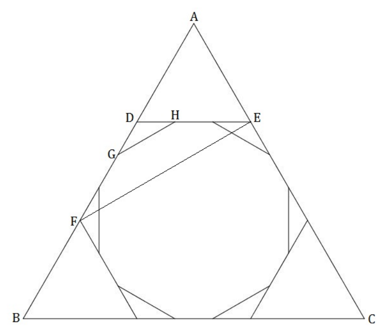
Hence $p$ must divides 30. The only possibility is $p=5$. On checking we see that $p=5, q=3$ indeed satisfies the equation. □

4. Note $10^{200}$ has 201 digits. Then $x$ must have atmost 200 digits. If $x$ has 199 digits then $y$ have atmost 199 digits. Hence their sum cannot be a 201 digit number. Thus $x$ has exactly 200 digits. Let $x_1 x_2 \cdots x_{200}$ be the decimal representation of $x$ and $y_1 y_2 \cdots y_{200}$ be the decimal representation of $y$. Suppose $x$ is not divisible by 10. Then $x_{200} \neq 0$. Thus $y_{200} = 10 - x_{200}$. Thus on adding the unit digits we carry 1 to the ten's digit. Thus $x_{199} + y_{199}$ must be 9. Again we carry 1 to the next digit and so on. Thus we arrive that $x_i + y_i = 9$ for all $i < 200$ and $x_{200} + y_{200} = 10$. Hence

$$x_1 + x_2 + \cdots + x_{200} + y_1 + y_2 + \cdots + y_{200} = 9 \times 199 + 10 = \text{odd}$$

But $y$ is just a rearrangement of $x$. Hence sum of digits of $x$ plus sum of digits of $y$ must be even. This gives us a contradiction. Hence $x$ is divisible by 10. □

5. We label some of the points of figure as shown below.




We join the diagonal of $FE$ of the hexagon $P_1$. Observe that $AE : AC = 1 : 3$ and $AD : AB = 1 : 3$ and hence $DE \parallel BC$. Thus $\triangle ADE$ and $\triangle ABC$ are similar. Hence by properties of similar triangles Area $ADE$ : Area $ABC = 1 : 9$. Hence Area $ADE = \frac{1}{9}$. Similarly area of other similar 'corners' are $\frac{1}{9}$. Hence area of hexagon is $1 - \frac{3}{9} = \frac{2}{3}$. We now focus on the corners inside the hexagon. Note that $DG : DF = 1 : 3$ and $DH : DE = 1 : 3$ and hence $GH \parallel FE$. Thus $\triangle DGH$ and $\triangle DFE$ are similar. Hence by properties of similar triangles Area $DGH$ : Area $DFE = 1 : 9$. Again $DE$ is the median of triangle $AEF$. Hence Area $DFE$ = Area $ADE = \frac{1}{9}$. Thus Area $DGH = \frac{1}{81}$. Similarly area of other 5 small corners are $\frac{1}{81}$. Hence Area of $P_2$ equals $\dfrac{2}{3} - \dfrac{6}{81} = \dfrac{2}{3} - \dfrac{2}{27} = \boxed{\dfrac{16}{27}}$ □

6. Only the fact that whether the numbers are odd(O) or even(E) is important for the problem. Note that at any stage if all the numbers are even, player 1 can never win. Player 2 target would be to leave two even number after penultimate move. Suppose we start with $OOEE$ sequence on board. If player 1 has any hope to win, he must not convert the odds into even. Hence the possible sequence after player 1 move is $OOE$ or $OEE$, then on adding first two terms or multiplying first two terms player 2 wins.

   For the general $OOOOEEEE$ problem, we will try to reach $OOEE$ or $EEEE$. For this 8-string, we either (i) try to maintain the initial symmetry of odds and evens or (ii) reduce the number of odds but keeping then separated from even numbers.

If in any step player 1 chooses 2 evens to modify, no of odds remain same, even numbers decreases by 1. Then player 2 can take 2 odd numbers and multiply them and reach case (i). Similar steps can be taken if player 1 takes 2 odds and multiplies them. If player 1 takes 2 odd numbers and get an even then two case can occur:

(a) There is an even number adjacent to the one player 1 got in the step; take two evens and do any operation

(b) There is an odd number adjacent to it, then the string has $EO$ in it. Transform it to $O$ It's easy to see this converts the previous string to strings mentioned in (i) and (ii),

Finally in the case player 1 chooses $OE$:

(a) $OE \to E$: We take two even and transform it to even

(b) $OE \to O$: We take two odd and transform them to odd.